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PRODUCTS OF POLYNOMIALS IN UNIFORM NORMS

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Abstract. We study inequalities connecting a product of uniform norms of polynomials with the norm of their product. This subject includes the well known Gel'fond-Mahler inequalities for the unit disk and Kneser inequality for the segment $[-1,1]$. Using tools of complex analysis and potential theory, we prove a sharp inequality for norms of products of algebraic polynomials over an arbitrary compact set of positive logarithmic capacity in the complex plane. The above classical results are contained in our theorem as special cases.

It is shown that the asymptotically extremal sequences of polynomials, for which this inequality becomes an asymptotic equality, are characterized by their asymptotically uniform zero distributions. We also relate asymptotically extremal polynomials to the classical polynomials with asymptotically minimal norms.

1. Introduction

Let E be a compact set in the complex plane $\mathbb{C}$. For a continuous function f on E, we define the uniform norm on E as follows:

$$\|f\|_E = \max_{z\in E} |f(z)|.$$

Consider algebraic polynomials $\{p_k(z)\}_{k=1}^m$ of one complex variable and their product

$$p(z) := \prod_{k=1}^m p_k(z).$$

This paper is devoted to a study of polynomial inequalities of the form

$$\prod_{k=1}^m \|p_k\|_E \le C\|p\|_E. \tag{1.1}$$

In particular, if $\deg p = n$ is the degree of the product polynomial $p(z)$, then we are interested in the asymptotic behavior of the constant C, as $n \to \infty$.

While the inequality opposite to (1.1) is obvious with $C = 1$, (1.1) itself has been studied in a number of papers, by considering various cases of the set E and

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different aspects of (1.1). Apparently, one of the first results in this direction is due to Kneser [15], for $E = [-1, 1]$ and $m = 2$ (see also Aumann [1]), who proved that

$$\|p_1\|_{[-1,1]}\|p_2\|_{[-1,1]} \le K_{\ell,n}\|p_1p_2\|_{[-1,1]}, \tag{1.2}$$

where

$$K_{\ell,n} := 2^{n-1}\prod_{k=1}^{\ell}\left(1+\cos\frac{2k-1}{2n}\pi\right)\prod_{k=1}^{n-\ell}\left(1+\cos\frac{2k-1}{2n}\pi\right), \tag{1.3}$$

$\deg p_1 = \ell$ and $\deg(p_1p_2) = n$. Note that (1.2) becomes an equality for the Chebyshev polynomial $t(z) = \cos n \arccos z = p_1(z)p_2(z)$, with a proper choice of the factors $p_1(z)$ and $p_2(z)$. P. B. Borwein [7] has recently given a new proof of (1.2)-(1.3) and generalized this to the multifactor inequality

$$\prod_{k=1}^{m}\|p_k\|_{[-1,1]} \le 2^{n-1}\prod_{k=1}^{[\frac{n}{2}]}\left(1+\cos\frac{2k-1}{2n}\pi\right)^2\|p\|_{[-1,1]}. \tag{1.4}$$

He has also shown that

$$2^{n-1}\prod_{k=1}^{[\frac{n}{2}]}\left(1+\cos\frac{2k-1}{2n}\pi\right)^2 \sim (3.20991\ldots)^n, \text{ as } n\to\infty. \tag{1.5}$$

Another case of the inequality (1.1) was considered by Gel'fond [13, p. 135] in connection with the theory of transcendental numbers, for $E = \overline{D}$, where $D := \{w : |w| < 1\}$ is the unit disk,

$$\prod_{k=1}^{m}\|p_k\|_{\overline{D}} \le e^n\|p\|_{\overline{D}}. \tag{1.6}$$

The latter inequality was improved by Mahler [18], who replaced e by 2:

$$\prod_{k=1}^{m}\|p_k\|_{\overline{D}} \le 2^n\|p\|_{\overline{D}}. \tag{1.7}$$

It is easy to see that the base 2 cannot be decreased, if $m = n$ and $n \to \infty$. However, (1.7) has recently been further improved in two directions. D. W. Boyd [8, 9] showed that, by taking into account the number of factors m in (1.7), one has

$$\prod_{k=1}^{m}\|p_k\|_{\overline{D}} \le (C_m)^n\|p\|_{\overline{D}}, \tag{1.8}$$

where

$$C_m := \exp\left(\frac{m}{\pi}\int_0^{\pi/m}\log\left(2\cos\frac{t}{2}\right)dt\right) \tag{1.9}$$

is asymptotically best possible for each fixed m, as $n \to \infty$. Kroó and Pritsker [16] showed that, for any $m \le n$,

$$\prod_{k=1}^{m}\|p_k\|_{\overline{D}} \le 2^{n-1}\|p\|_{\overline{D}}, \tag{1.10}$$

where equality holds in (1.10) for *each* $n \in \mathbb{N}$, with $m = n$ and $p(z) = z^n - 1$.

The above-mentioned results represent only a selection, which is directly related to the subject of this paper, from the existing literature on inequalities for products of polynomials in various norms. Another particularly important direction is related to polynomials in many variables. We do not discuss it here, but give the references to the results of Mahler [19] for the polydisk, of Avanissian and Mignotte [2] for the unit ball in $\mathbb{C}^k$, of Beauzamy and Enflo [4], and of Beauzamy, Bombieri, Enflo and Montgomery [3] for an extensive study of products of polynomials in several variables, using different norms. We would also like to mention the research on norms of products of polynomials in abstract Banach spaces by Benitez, Sarantopoulos and Tonge [5].

The rest of this paper is organized as follows. Our general results are stated below in Section 2. Their applications for the unit disk, the segment $[-1,1]$ and a circular arc are given in Section 3. We discuss a question about the possibility of improvement for a fixed number of factors in Section 4. The proofs of the results stated in Sections 2, 3 and 4 can be found in Section 5.

2. General results

The known results discussed in the Introduction indicate that the constant C in (1.1) typically grows exponentially fast with n, which is the degree of the product, and it also depends on the set E, as one might expect. Therefore, we consider a general problem of finding the *smallest* constant $M_E > 0$, such that

$$\prod_{k=1}^{m} \|p_k\|_E \le M_E^n \|p\|_E \tag{2.1}$$

for arbitrary algebraic polynomials $\{p_k(z)\}_{k=1}^m$ with complex coefficients, where $p(z) = \prod_{k=1}^m p_k(z)$ and $n = \deg p$, as before.

In order to give a solution of the above problem, we have to introduce certain notions from the logarithmic potential theory (cf. [24]). Let $\text{cap}(E)$ be the *logarithmic capacity* of a compact set $E \subset \mathbb{C}$. For E with $\text{cap}(E) > 0$, denote the *equilibrium measure* of E (in the sense of the logarithmic potential theory) by μ_E. We remark that μ_E is a positive unit Borel measure supported on E, $\text{supp}\mu_E \subset E$ (see [24, p. 55]). Define

$$d_E(z) := \max_{t \in E} |z - t|, \qquad z \in \mathbb{C}, \tag{2.2}$$

which is clearly a positive and continuous function on $\mathbb{C}$.

Theorem 2.1. *Let $E \subset \mathbb{C}$ be a compact set,* $\text{cap}(E) > 0$*. Then the best constant M_E in* (2.1) *is given by*

$$M_E = \frac{\exp\left(\int \log d_E(z) d\mu_E(z)\right)}{\text{cap}(E)}. \tag{2.3}$$

We show in the next section that this general result gives the inequality (1.7) of Mahler and the asymptotic version of Borwein-Kneser inequality (1.4)-(1.5).

Note that the restriction $\text{cap}(E) > 0$ excludes only very *thin* sets from our consideration (see [24, pp. 63-66]), e.g., finite sets in the plane. But if E consists of finitely many points (more than one) then the inequality (2.1) (or (1.1)) cannot be true at all, which is easy to see for a polynomial $p(z)$ with zeros at every point of E and for its linear factors $\{p_k(z)\}_{k=1}^n$. On the other hand, Theorem 2.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [24, p. 56]). In particular, if E is a continuum, then we obtain the following crude but interesting estimate.

Corollary 2.2. *Let $E \subset \mathbb{C}$ be a bounded continuum (not a single point). Then we have*

$$M_E \leq \frac{\operatorname{diam}(E)}{\operatorname{cap}(E)} \leq 4, \tag{2.4}$$

where $\operatorname{diam}(E)$ *is the Euclidean diameter of the set* E.

An interesting question connected with our problem is about the nature of the extremal polynomials for (2.1), i.e., about those polynomials, for which (2.1) becomes an asymptotic equality, as $n \to \infty$. The classical cases $E = \overline{D}$ and $E = [-1, 1]$ suggest that the extremal polynomials have certain very special properties, and special zero distributions, in particular. We show that this is true in general. Namely, we say that a sequence of polynomials $\{Q_n(z)\}_{n=1}^{\infty}$, $\deg Q_n = n$, is asymptotically extremal for (2.1) if

$$\lim_{n\to\infty} \left(\frac{\prod_{k=1}^{n} \|q_{k,n}\|_E}{\|Q_n\|_E} \right)^{1/n} = M_E, \tag{2.5}$$

where $\{q_{k,n}(z)\}_{k=1}^{n}$ are the linear factors of $Q_n(z)$, i.e.,

$$Q_n(z) =: \prod_{k=1}^{n} q_{k,n}(z), \quad n \in \mathbb{N}. \tag{2.6}$$

Observe that there is no loss of generality if we consider only *monic* polynomials $\{p_k(z)\}_{k=1}^{m}$ in (2.1), so that their product $p(z)$ is also monic. In particular, if we have a sequence of asymptotically extremal polynomials $\{Q_n(z)\}_{n=1}^{\infty}$, then the corresponding sequence of monic polynomials, obtained by dividing each $Q_n(z)$ by its leading coefficient, is also asymptotically extremal. Recall that for any monic polynomial $P(z)$ of degree n, we have

$$\|P\|_E \geq (\operatorname{cap}(E))^n, \tag{2.7}$$

where $E \subset \mathbb{C}$ is an arbitrary compact set (cf. Theorem 5.5.4(a) in [21, p. 155]). Thus, if a sequence of monic polynomials $\{P_n(z)\}_{n=1}^{\infty}$, $\deg P_n = n$, satisfies

$$\lim_{n\to\infty} \|P_n\|_E^{1/n} = \operatorname{cap}(E), \tag{2.8}$$

then it is customary to say that such polynomials have asymptotically minimal norms on E. Sequences of polynomials with asymptotically minimal norms have been studied in many papers, e.g., see Faber [11], Fekete and Walsh [12], Widom [25, 26], Blatt, Saff and Simkani [6], Mhaskar and Saff [20], Stahl and Totik [23]. Our next result relates monic asymptotically extremal polynomials for (2.1) to monic polynomials satisfying (2.8).

Theorem 2.3. *Let $\{Q_n(z)\}_{n=1}^{\infty}$ be a sequence of monic asymptotically extremal polynomials for* (2.1) *on a compact set $E \subset \mathbb{C}$, $\operatorname{cap}(E) > 0$. If all zeros of $Q_n(z)$ are uniformly bounded for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \|Q_n\|_E^{1/n} = \operatorname{cap}(E). \tag{2.9}$$

We remark that if the condition that all zeros be uniformly bounded is dropped, then (2.9) may not always be true. This is easy to see for $E = \overline{D}$ and $\tilde{Q}_n(z) :=$

$(z^{n-1}+1)(z+10^n)$, $n \in \mathbb{N}$. Since $M_{\overline{D}} = 2$ (see [18] or Section 3.1 of this paper), equation (2.5) is readily checked for this sequence. However,

$$\lim_{n\to\infty} \|\tilde{Q}_n\|_E^{1/n} = 10 \neq \operatorname{cap}(\overline{D}) = 1.$$

Also, note that polynomials with asymptotically minimal norms need not be asymptotically extremal for (2.1), in general. To verify this, it is sufficient to consider $P_n(z) = z^n$, $n \in \mathbb{N}$, and $E = \overline{D}$.

We next turn to the study of asymptotic zero distributions of asymptotically extremal polynomials (not necessarily monic).

Assume that $\{z_{k,n}\}_{k=1}^n \subset \mathbb{C}$ are the zeros of $Q_n(z)$ and define the *normalized zero counting measure* for $Q_n(z)$ by

$$\nu_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{z_{k,n}}, \quad n \in \mathbb{N}, \tag{2.10}$$

where δ_z denotes the Dirac measure at z. Note that ν_n is a unit Borel measure and $\operatorname{supp}\nu_n = \{z_{k,n}\}_{k=1}^n$, $n \in \mathbb{N}$. A sequence of Borel measures $\{\mu_n\}_{n=1}^\infty$ is said to converge to a Borel measure μ in the *weak* topology* if

$$\lim_{n\to\infty} \int f(z)d\mu_n(z) = \int f(z)d\mu(z) \tag{2.11}$$

for any continuous function $f(z)$ with compact support in $\mathbb{C}$. We write in this case that $\mu_n \xrightarrow{*} \mu$, as $n \to \infty$.

Theorem 2.4. *Let $E \subset \mathbb{C}$ be a compact set and let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus E$. Suppose that E satisfies one of the following:*
(i) E has empty interior, $\overline{\mathbb{C}} \setminus E = \Omega$ and Ω is regular;
(ii) $E = \overline{G}$, where G is a bounded domain, and $\mathbb{C} \setminus \overline{\Omega}$ is connected.
If $\{Q_n(z)\}_{n=1}^\infty$ is an asymptotically extremal sequence of polynomials, then we have for the normalized zero counting measures ν_n of (2.10) *that*

$$\nu_n \xrightarrow{*} \mu_E, \qquad \text{as } n \to \infty. \tag{2.12}$$

It is well known (cf. [24, p. 79]) that the equilibrium measure μ_E is supported on the *outer boundary* of E, i.e., on the boundary of the unbounded component Ω of $\overline{\mathbb{C}} \setminus E$. Thus, Theorem 2.4 says that the zeros of asymptotically extremal polynomials are asymptotically equidistributed along the outer boundary of E, according to μ_E, as $n \to \infty$. Classical examples of the asymptotically extremal polynomials with the above zero distributions include Fekete polynomials, Leja polynomials, Chebyshev polynomials for sets with empty interior, etc.

In fact, Theorem 2.4 allows a converse stated below.

Theorem 2.5. *Let $E \subset \mathbb{C}$ be a compact set,* $\operatorname{cap}(E) > 0$*, and let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus E$. If Ω is regular, then* (2.12) *implies that the sequence of polynomials $\{Q_n(z)\}_{n=1}^\infty$ is asymptotically extremal.*

For the notion of regularity in the sense of Dirichlet problem, we refer to [21, Ch. 4]. One can see that the regularity of Ω is essential in the above theorem, by considering the following example. Let $E = \overline{D} \cup \{a\}$, where $a > 3$ is real, so that a is the isolated irregular point. Then $\operatorname{cap}(\overline{D} \cup \{a\}) = \operatorname{cap}(\overline{D}) = 1$ and

$\mu_{\overline{D}\cup\{a\}} = \mu_{\overline{D}} = \frac{1}{2\pi}d\theta$ (see Theorems III.31 and III.37 in [24]), where $d\theta$ is the arclength on ∂D. This gives

$$M_{\overline{D}\cup\{a\}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log d_{\overline{D}\cup\{a\}}(e^{i\theta})d\theta\right) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta}-a|d\theta\right) = a.$$

For the polynomials $\tilde{Q}_n(z) = z^n - 1$, $n \in \mathbb{N}$, we have that

$$\nu_n(\tilde{Q}_n) \xrightarrow{*} \frac{1}{2\pi}d\theta = \mu_{\overline{D}} = \mu_{\overline{D}\cup\{a\}}, \quad \text{as } n \to \infty.$$

On the other hand, using the above weak* convergence, we obtain

$$\lim_{n\to\infty}\left(\frac{\prod_{k=1}^n \|z-e^{i\frac{2\pi}{n}k}\|_{\overline{D}\cup\{a\}}}{\|z^n-1\|_{\overline{D}\cup\{a\}}}\right)^{1/n} = \frac{1}{a}\lim_{n\to\infty}\left(\prod_{k=1}^n |a-e^{i\frac{2\pi}{n}k}|\right)^{1/n}$$
$$= \frac{1}{a}\lim_{n\to\infty}\exp\left(\int \log|a-z|d\nu_n(\tilde{Q}_n)(z)\right) = \frac{1}{a}\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|a-e^{i\theta}|d\theta\right) = 1,$$

i.e., the sequence $\{\tilde{Q}_n(z)\}_{n=1}^\infty$ is not asymptotically extremal on $E = \overline{D}\cup\{a\}$.

3. Applications

We consider here three special cases of the general results of Section 2, where the measure μ_E is known explicitly, and obtain the explicit values of M_E. Those are the cases of the unit disk, of the segment $[-1, 1]$ and of a circular arc. Further, we give several known representations of μ_E for general sets and discuss how to obtain the explicit form of μ_E from them.

3.1. **Unit Disk.** For the unit disk $D = \{w : |w| < 1\}$, we have that $\text{cap}(\overline{D}) = 1$ [24, p. 84] and that

$$\mu_{\overline{D}} = \frac{1}{2\pi}d\theta, \tag{3.1}$$

where $d\theta$ is the arclength on ∂D. Thus, Theorem 2.1 yields

$$M_{\overline{D}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log d_{\overline{D}}(e^{i\theta})\ d\theta\right) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log 2\ d\theta\right) = 2, \tag{3.2}$$

so that we immediately obtain Mahler's inequality (1.7).

The results of Theorems 2.4 and 2.5 are apparently new even in this classical case.

Corollary 3.1. *A sequence of polynomials $\{Q_n(z)\}_{n=1}^\infty$ is asymptotically extremal for* (2.1) *on $E = \overline{D}$ if and only if*

$$\nu_n \xrightarrow{*} \frac{1}{2\pi}d\theta, \quad \text{as } n\to\infty, \tag{3.3}$$

where ν_n is defined by (2.10) *and where $d\theta$ is the arclength on ∂D.*

3.2. **Segment** $[-1,1]$**.** If $E=[-1,1]$, then $\mathrm{cap}([-1,1])=1/2$ and

$$\mu_{[-1,1]}=\frac{dx}{\pi\sqrt{1-x^2}},\quad x\in[-1,1], \tag{3.4}$$

which is the Chebyshev (or arcsin) distribution (see [24, p. 84]). Using Theorem 2.1, we obtain

$$\begin{aligned} M_{[-1,1]} &= 2\exp\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\log d_{[-1,1]}(x)}{\sqrt{1-x^2}}dx\right)=2\exp\left(\frac{2}{\pi}\int_0^1\frac{\log(1+x)}{\sqrt{1-x^2}}dx\right)\\ &= 2\exp\left(\frac{2}{\pi}\int_0^{\pi/2}\log(1+\sin t)dt\right)\approx 3.2099123, \end{aligned}\tag{3.5}$$

which gives the asymptotic version of Borwein's inequality (1.4)-(1.5). Again, the description of the extremal polynomials appears to be new in this case.

Corollary 3.2. *A sequence of polynomials* $\{Q_n(z)\}_{n=1}^\infty$ *is asymptotically extremal for* (2.1) *on* $E=[-1,1]$ *if and only if*

$$\nu_n\xrightarrow{*}\frac{dx}{\pi\sqrt{1-x^2}},\quad x\in[-1,1],\ \textit{as } n\to\infty, \tag{3.6}$$

where ν_n *is defined by* (2.10).

3.3. **Circular Arc.** We now consider the case $E=\gamma$, where $\gamma:=\{e^{i\theta}:|\theta|\le\alpha/2<\pi\}$. It is known that

$$\mathrm{cap}(\gamma)=\sin(\alpha/4) \tag{3.7}$$

and that

$$\Phi(z)=\frac{z-1+\sqrt{(z-e^{i\alpha/2})(z-e^{-i\alpha/2})}}{2\sin(\alpha/4)},\quad z\in\overline{\mathbb{C}}\setminus\gamma, \tag{3.8}$$

is the conformal mapping of $\overline{\mathbb{C}}\setminus\gamma$ onto $\overline{\mathbb{C}}\setminus\overline{D}$, such that $\Phi(\infty)=\infty$ and $\Phi'(\infty)=\sin(\alpha/4)$ (cf. [21, p. 137]). Note that every point $z=e^{i\theta}\in\gamma$, except for the endpoints of γ, has two images on $\{w:|w|=1\}$:

$$\Phi_+(e^{i\theta})=\frac{e^{i\theta}-1+e^{i\theta/2}\sqrt{2(\cos\theta-\cos(\alpha/2))}}{2\sin(\alpha/4)}, \tag{3.9}$$

and

$$\Phi_-(e^{i\theta})=\frac{e^{i\theta}-1-e^{i\theta/2}\sqrt{2(\cos\theta-\cos(\alpha/2))}}{2\sin(\alpha/4)}, \tag{3.10}$$

which correspond to the two branches of the root in (3.8). We show in Section 5 that

$$\mu_\gamma=\frac{|\Phi_+(e^{i\theta})+\sin(\alpha/4)|+|\Phi_-(e^{i\theta})+\sin(\alpha/4)|}{2\pi\sqrt{2(\cos\theta-\cos(\alpha/2))}}d\theta,\quad e^{i\theta}\in\gamma. \tag{3.11}$$

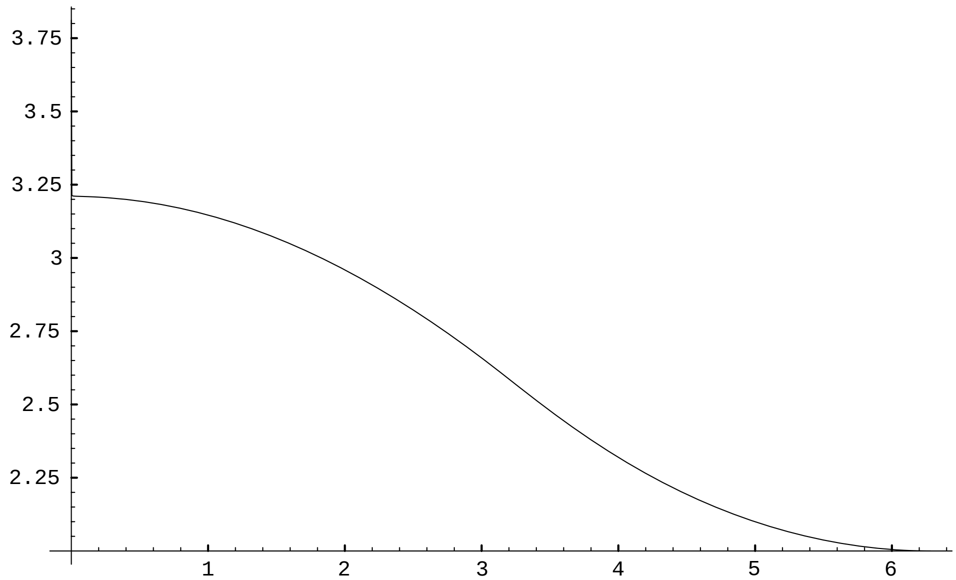

FIGURE 1. $M_\gamma(\alpha)$ for the circular arc $\gamma = \{e^{i\theta} : \ |\theta| \le \alpha/2 \le \pi\}$.

Corollary 3.3. *Let $E = \gamma = \{e^{i\theta} : |\theta| \le \alpha/2 < \pi\}$. Then* (2.1) *holds with*

(3.12)

$$M_\gamma(\alpha) = \begin{cases} \frac{1}{\sin(\alpha/4)} \exp\left(2\int_0^{\alpha/2} \log\left(2\sin(\theta/2+\alpha/4)\right) d\mu_\gamma(\theta)\right), & \alpha \in [0,\pi], \\ \frac{1}{\sin(\alpha/4)} \exp\left(2\int_0^{\pi-\alpha/2} \log\left(2\sin(\theta/2+\alpha/4)\right) d\mu_\gamma(\theta)\right. & \\ \qquad\qquad \left. +2\int_{\pi-\alpha/2}^{\alpha/2} \log 2 \ d\mu_\gamma(\theta)\right), & \alpha \in [\pi, 2\pi]. \end{cases}$$

Furthermore, a sequence of polynomials $\{Q_n(z)\}_{n=1}^\infty$ is asymptotically extremal for (2.1) *on $E = \gamma$ if and only if*

$$\nu_n \xrightarrow{*} \mu_\gamma, \qquad \text{as } n \to \infty, \tag{3.13}$$

where ν_n is defined by (2.10) *and μ_γ is defined by* (3.11).

The graph of $M_\gamma(\alpha)$, $\alpha \in [0, 2\pi]$, is given in Figure 1.

3.4. **General Sets.** We give several well known representations for the equilibrium measure μ_E in this subsection. First, consider the case of an arbitrary compact set $E \subset \mathbb{C}$, with $\mathrm{cap}(E) > 0$ and with the unbounded component of its complement $\overline{\mathbb{C}} \setminus E$ denoted by Ω. Then

$$\mu_E = \omega(\infty, \cdot, \Omega), \tag{3.14}$$

where $\omega(\infty, \cdot, \Omega)$ is the harmonic measure of Ω at ∞ (cf. [21, p. 105]). This useful identity implies that μ_E is invariant under certain conformal mappings of Ω and gives the following form of μ_E, especially important for our applications.

Assume that E is a closure of Jordan domain with rectifiable boundary. Since Ω is simply connected in this case, there exists a conformal mapping $\Phi : \Omega \to D' :=$

$\{w : |w| > 1\}$ normalized by

$$\Phi(\infty) = \infty \quad \text{and} \quad \lim_{z\to\infty} \frac{\Phi(z)}{z} = \frac{1}{\text{cap}(E)}, \tag{3.15}$$

(see [21, p. 133]). Furthermore, for any Borel set $B \subset \mathbb{C}$ we have from (3.14) that

$$\mu_E(B) = m(\Phi(B \cap \partial\Omega)), \tag{3.16}$$

where $dm = d\theta/(2\pi)$ is the normalized arclength on $\{|w| = 1\}$. This gives that

$$\mu_E(z) = \frac{1}{2\pi}|\Phi'(z)||dz|, \quad z \in \partial\Omega, \tag{3.17}$$

so that both $\text{cap}(E)$ and μ_E can be found from the conformal mapping Φ by (3.15) and (3.17). Thus, if an explicit form of Φ is known, then the constant M_E of (2.3) is also known explicitly. In fact, the cases of the unit disk, of the segment $[-1, 1]$ and of a circular arc, considered in the previous three subsections, are handled in a similar way.

Yet finding the capacity and the equilibrium measure of a general set E is a hard classical problem, often without an explicit solution. Fortunately, one can approach the problem of finding the constant M_E numerically, using sequences of asymptotically extremal polynomials. Many examples of suitable sequences of polynomials can be derived with the help of Theorem 2.5, by generating polynomials with asymptotically uniformly distributed zeros, such as Fekete polynomials, Leja polynomials, etc. We shall consider the numerical aspect of the problem elsewhere.

4. Possibility of improvement in a fixed number of factors case

We already mentioned a result of Boyd [9] (see (1.8)-(1.9) in the Introduction), that one can improve the constant in (2.3) for $E = \overline{D}$, by considering a *fixed number of factors* m in (2.1). On the other hand, comparing the results of Kneser (1.2)-(1.3) and of Borwein (1.4), one can immediately observe that the constant is the same for any $m \geq 2$, i.e., there is *no improvement* for $E = [-1, 1]$. The nature of this phenomenon can be easily explained by the presence of two endpoints in the case $E = [-1, 1]$. We give a more general result below.

It follows from the proof of Theorem 2.1 (cf. (5.26)), that

$$\prod_{k=1}^{m} \|p_k\|_E \leq \left(\frac{\exp(\int u_m(z) d\mu_E(z))}{\text{cap}(E)} \right)^n \|p\|_E \tag{4.1}$$

where

$$u_m(z) := \max_{1\leq k\leq m} \log|z - c_k| \tag{4.2}$$

and

$$|p_k(c_k)| = \|p_k\|_E, \quad k = 1, \ldots, m. \tag{4.3}$$

This is a generalization of Boyd's ideas in [9]. In fact, the value of the constant C_m in (1.9) is obtained in this way, see the proof of Theorem 1 in [9]. However, if $E = [-1, 1]$ then

$$\log d_{[-1,1]}(z) = \max(\log|z - 1|, \log|z + 1|), \quad z \in [-1, 1].$$

Consequently, $u_m(z) = \log d_{[-1,1]}(z)$ for any $z \in [-1, 1]$, if the set $\{c_k\}_{k=1}^m$ contains the endpoints $\{1, -1\}$. Thus, the difference between the constants in (4.1) and in (2.3) is essentially the difference between $u_m(z)$ and $\log d_E(z)$.

Consider the Fekete polynomials $\{F_n(z)\}_{n=1}^{\infty}$, $\deg F_n = n$, for the set E (cf. [21, p. 155]).

Theorem 4.1. *Let $E \subset \mathbb{C}$ be a compact set, $\operatorname{cap}(E) > 0$. Suppose that there exist points $\{\zeta_l\}_{l=1}^{s}$ such that*

$$d_E(z) = \max_{1\le l\le s} |z-\zeta_l|, \quad for\ any\ z \in \partial E. \tag{4.4}$$

If $m \ge s$ then we can find such factoring for the sequence of Fekete polynomials

$$F_n(z) = \prod_{k=1}^{m} F_{k,n}(z), \quad n \in \mathbb{N}, \tag{4.5}$$

that

$$\lim_{n\to\infty} \left(\frac{\prod_{k=1}^{m} \|F_{k,n}\|_E}{\|F_n\|_E} \right)^{1/n} = M_E. \tag{4.6}$$

Consequently, no improvement is possible in (2.1)*, for a fixed number of factors $m \ge s$, as $n \to \infty$.*

It is easy to see from the above theorem that there is no improvement in constant, for any $m \ge 2$, for such sets as a circular arc of angular measure at most π and a segment. Also, there is no improvement for any polygon with s vertices, if $m \ge s$.

5. Proofs

We start with several lemmas necessary for the proofs of our general results from Section 2. The following lemma is a generalization of Lemma 2 in [9].

Lemma 5.1. *Let $F \subset \mathbb{C}$ be a compact set (not a single point) and let*

$$d_F(z) := \max_{t\in F} |z-t|, \quad z \in \mathbb{C}.$$

Then $\log d_F(z)$ is a subharmonic function in $\mathbb{C}$ and

$$\log d_F(z) = \int \log|z-t|\,d\sigma(t), \quad z \in \mathbb{C}, \tag{5.1}$$

where σ is a positive unit Borel measure in $\mathbb{C}$ with unbounded support, i.e.,

$$\sigma(\mathbb{C}) = 1 \quad and \quad \infty \in \operatorname{supp}\sigma. \tag{5.2}$$

Furthermore, if $F = \overline{G}$, where G is a bounded domain, then

$$\operatorname{supp}\sigma \cap G \ne \emptyset. \tag{5.3}$$

Note that Lemma 5.1 reduces back to Lemma 2 of Boyd [9] in the case $F = \{c_k\}_{k=1}^{m}$, where $c_k, k = 1, \dots, m$ $(m \ge 2)$, are complex numbers and

$$\log d_{\{c_k\}_{k=1}^{m}}(z) = \max_{1\le k\le m} \log|z-c_k| =: u_m(z). \tag{5.4}$$

Proof. Note that

$$u(z) := \log d_F(z) = \sup_{t\in F} \log|z-t|, \quad z \in \mathbb{C}.$$

Since $\log|z-t|$ is upper semicontinuous on $\mathbb{C} \times F$ and is subharmonic in z for each $t \in F$, Theorem 2.4.7 of [21, p. 38] implies that $u(z)$ is subharmonic in $\mathbb{C}$.

Let $\{c_k\}_{k=1}^{\infty} \subset \partial F$ be a sequence of points dense in ∂F. On defining

$$u_m(z) := \max_{1\le k\le m} \log|z-c_k|, \quad z\in\mathbb{C},\ m\ge 2, \tag{5.5}$$

we observe that $\{u_m(z)\}_{m=1}^{\infty}$ is a monotonically increasing sequence of continuous subharmonic functions in $\mathbb{C}$, such that

$$u(z) = \lim_{m\to\infty} u_m(z), \quad z\in\mathbb{C}, \tag{5.6}$$

where the above convergence is uniform on compact subsets of $\mathbb{C}$. It follows from Lemma 2 of [9] that, for any $m\ge 2$,

$$u_m(z) = \int \log|z-t|d\sigma_m(t), \quad z\in\mathbb{C}, \tag{5.7}$$

where σ_m is a probability measure in $\mathbb{C}$. Let $D_R := \{z : |z| < R\},\ R > 2$. Then

$$u_m(z) = \int_{\overline{D_R}} \log|z-t|d\sigma_m(t) + h_m(z), \quad z\in\mathbb{C}, \tag{5.8}$$

where

$$h_m(z) = \int_{\mathbb{C}\setminus\overline{D_R}} \log|z-t|d\sigma_m(t), \quad z\in\mathbb{C}, \tag{5.9}$$

is harmonic in D_R for any $m\ge 2$. It follows from (5.8) that

$$\begin{aligned}\inf_{z\in D_R} h_m(z) &\ge \inf_{z\in D_R} u_m(z) - \sup_{z\in D_R}\int_{\overline{D_R}} \log|z-t|d\sigma_m(t)\\ &\ge \inf_{z\in D_R} u_2(z) - \log(2R) =: A_R, \quad m\ge 2,\end{aligned}$$

because $\sigma_m(\mathbb{C}) = 1$. Since $\{h_m(z) - A_R\}_{m=2}^{\infty}$ is a sequence of positive harmonic functions in D_R, we have that either $h_m(z)\to+\infty$ locally uniformly in D_R or there is a subsequence $N_1\subset\mathbb{N}$ such that $h_m(z),\ m\in N_1$, converges locally uniformly to a harmonic function $\tilde{h}_R(z)$ in D_R (see Theorem 1.3.10 in [21, p. 16]). We show that the first case is impossible, in fact. Indeed, if that were possible, then

$$\lim_{m\to\infty}\int_{\overline{D_R}} \log|z-t|d\sigma_m(t) = -\infty, \quad z\in D_R, \tag{5.10}$$

by (5.6) and (5.8), where the above divergence is locally uniform in D_R. Using (5.10), submean inequality and Fubini's theorem, we have

$$\begin{aligned}-\infty &= \lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\int_{\overline{D_R}}\log|Re^{i\theta}/2-t|d\sigma_m(t)d\theta\\ &\ge \lim_{m\to\infty}\int_{\overline{D_R}}\left(\frac{1}{2\pi}\int_0^{2\pi}\log|Re^{i\theta}/2-t|d\theta\right)d\sigma_m(t)\\ &\ge \lim_{m\to\infty}\sigma_m(\overline{D_R})\log\frac{R}{2}\ge 0,\end{aligned}$$

which is an obvious contradiction. Next, we choose a subsequence $N_2\subset N_1$, such that

$$\sigma_m|_{\overline{D_R}} \xrightarrow{*} \tilde{\sigma}_R, \quad \text{as } m\to\infty,\ m\in N_2, \tag{5.11}$$

by Helley's selection theorem (see Theorem 0.1.2 in [22]). Note that $\operatorname{supp}\tilde{\sigma}_R \subset \overline{D_R}$, $\tilde{\sigma}_R(\mathbb{C}) \leq 1$, and

$$\limsup_{\substack{m\to\infty \\ m\in N_2}} \int_{\overline{D_R}} \log|z-t|d\sigma_m(t) = \int \log|z-t|d\tilde{\sigma}_R$$

quasi everywhere (with a possible exception of a set of zero capacity) in $\mathbb{C}$, by Theorem I.6.9 of [22]. Thus, passing to the limit in (5.8), as $m \to \infty$, $m \in N_2$, we have

$$u(z) = \int \log|z-t|d\tilde{\sigma}_R(t) + \tilde{h}_R(z) \tag{5.12}$$

holds quasi everywhere in D_R.

Using Riesz Decomposition Theorem [21, p. 76] for $u(z)$ in D_R, we obtain

$$u(z) = \int \log|z-t|d\sigma_R(t) + h_R(z), \quad z \in D_R, \tag{5.13}$$

where $h_R(z)$ is harmonic in D_R and σ_R is supported in D_R. Hence, the Unicity Theorem (cf. Theorem II.2.1 in [22]), (5.12) and (5.13) imply that

$$\sigma_R = \tilde{\sigma}_R|_{D_R} \quad \text{and} \quad \sigma_R(\mathbb{C}) \leq 1, \tag{5.14}$$

for any $R > 2$. Clearly, the sequence of measures σ_R is monotonically increasing, as $R \to \infty$, to the associated Riesz measure σ for $u(z)$ in $\mathbb{C}$ (cf. [10, p. 51]). Observe that $\sigma(\mathbb{C}) \leq 1$, by (5.14) and (5.11), and that

$$\lim_{z\to\infty}(u(z) - \log|z|) = \lim_{z\to\infty} \log\frac{d_F(z)}{|z|} = 0.$$

Let us show that $\sigma(\mathbb{C}) = 1$. Integrating (5.13), we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} u((R-\epsilon)e^{i\theta})d\theta \\
=\ &\frac{1}{2\pi}\int_0^{2\pi}\left(\int_{D_R} \log|(R-\epsilon)e^{i\theta} - t|d\sigma(t) + h_R((R-\epsilon)e^{i\theta})\right)d\theta \\
=\ &\int_{D_R}\left(\frac{1}{2\pi}\int_0^{2\pi} \log|(R-\epsilon)e^{i\theta} - t|d\theta\right)d\sigma(t) + h_R(0) \\
=\ &\sigma(\overline{D_{R-\epsilon}})\log(R-\epsilon) + \int_{D_R\setminus\overline{D_{R-\epsilon}}} \log|t|d\sigma(t) + h_R(0),
\end{aligned}$$

which gives on letting $\epsilon \to 0$ that

$$\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})d\theta - \sigma(D_R)\log R = h_R(0).$$

If $\sigma(\mathbb{C}) < 1$ then, passing to the limit in the above equation, as $R \to \infty$, we have that

$$\lim_{R\to\infty} h_R(0) = +\infty. \tag{5.15}$$

Note that $\int \log|t|d\sigma_R(t)$ is increasing with R, as $\log|t| > 0$ for $|t| > 1$. Considering (5.13) for $z = 0$, we deduce that $h_R(0)$ is decreasing when $R \to \infty$, which contradicts (5.15). Thus, $\sigma(\mathbb{C}) = 1$ and

$$\limsup_{R\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})d\theta - \sigma(\overline{D_R})\log R\right) \geq 0.$$

It follows from the above inequality and (9.3) of Theorem 1.IV.9(d) in [10, p. 53] that

$$u(z) = \int \log|z-t|d\sigma(t) + h(z), \quad z \in \mathbb{C},$$

where $h(z)$ is a harmonic function in $\mathbb{C}$ given by

$$\begin{aligned} h(z) &= \lim_{R\to\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} u(z+Re^{i\theta})d\theta - \sigma(\mathbb{C}) \log R \right) \\ &= \lim_{R\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \log \frac{d_F(z+Re^{i\theta})}{R} d\theta = 0, \quad z \in \mathbb{C}. \end{aligned}$$

This completes the proof of (5.1).

Let us show that suppσ is unbounded. Assume to the contrary that it is compact, then $u(z)$ must be harmonic near infinity by (5.1). We can also assume, without loss of generality, that $0 \in F$ and define the function

$$v(z) := u(z) - \log|z| = \log d_F(z) - \log|z|, \quad z \in \mathbb{C}.$$

Observe that $v(z)$ is continuous for any $z \in \mathbb{C}$ with $|z| \geq R > 0$. Moreover, by assigning $v(\infty) = 0$ by continuity, we have that $v(z)$ is harmonic in $\overline{\mathbb{C}} \setminus D_R$, for R sufficiently large. It follows by the mean value property that

$$\frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})d\theta = v(\infty) = 0, \quad r > R.$$

On the other hand, $d_F(z) \geq |z|$ for any $z \in \mathbb{C}$ and $d_F(re^{i\theta}) > r$ for some $\theta \in [0, 2\pi)$, because F has other points beside $z = 0$. This implies that

$$\frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})d\theta > 0,$$

which is a direct contradiction.

We now turn to the proof of (5.3) in the case $F = \overline{G}$, where G is a bounded domain. Again we assume to the contrary that supp$\sigma \cap G = \emptyset$, which implies that $u(z)$ is harmonic in G by (5.1). Let $z \in G$ and $\zeta_z \in \partial G$ be such that

$$d_{\overline{G}}(z) = |z - \zeta_z|.$$

Then, for the harmonic function

$$w(t) := \log|t - \zeta_z| - \log d_{\overline{G}}(t), \quad t \in G,$$

we have

$$w(z) = 0 \quad \text{and} \quad w(t) \leq 0, \quad t \in F.$$

Consequently, $w(t)$ attains its maximum at $t = z \in G$, so that $w(t) \equiv 0$ in G by the maximum principle (see Theorem 1.1.8 in [21, p. 6]). But

$$d_{\overline{G}}(t) = |t - \zeta_z|, \quad t \in G,$$

is clearly impossible for $t \to \zeta_z$, $t \in G$. □

Lemma 5.2. (Bernstein-Walsh) *Let $E \subset \mathbb{C}$ be a compact set,* cap$(E) > 0$*, with the unbounded component of $\overline{\mathbb{C}} \setminus E$ denoted by Ω. Then, for any polynomial $p(z)$ of degree n, we have*

$$|p(z)| \leq \|p\|_E \, e^{ng_\Omega(z,\infty)}, \quad z \in \mathbb{C}, \tag{5.16}$$

where $g_\Omega(z,\infty)$ is the Green function of Ω, with pole at ∞, satisfying

$$g_\Omega(z,\infty) = \log \frac{1}{\operatorname{cap}(E)} + \int \log|z-t|d\mu_E(t), \quad z \in \mathbb{C}. \tag{5.17}$$

This is a well known result about the upper bound (5.16) for the growth of $p(z)$ off the set E (see [21, p. 156], for example). The representation (5.17) for $g_\Omega(z,\infty)$ is also classical (cf. Theorem III.37 in [24, p. 82]).

Consider the n-th Fekete points $\{a_{k,n}\}_{k=1}^n$ for a compact set $E \subset \mathbb{C}$ (cf. [21, p. 152]). Let

$$F_n(z) := \prod_{k=1}^{n} (z - a_{k,n}) \tag{5.18}$$

be the Fekete polynomial of degree n, and define the normalized counting measures in Fekete points by

$$\tau_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{a_{k,n}}, \quad n \in \mathbb{N}. \tag{5.19}$$

Lemma 5.3. *For a compact set $E \subset \mathbb{C}$, $\operatorname{cap}(E) > 0$, we have that*

$$\lim_{n\to\infty} \|F_n\|_E^{1/n} = \operatorname{cap}(E) \tag{5.20}$$

and

$$\tau_n \xrightarrow{*} \mu_E, \quad \text{as } n \to \infty. \tag{5.21}$$

Equation (5.20) is standard (see Theorems 5.5.4 and 5.5.2 in [21, pp. 153-155]), while (5.21) follows from (5.20) and Theorem 2.1 of [6].

Proof of Theorem 2.1. First we show that the best constant M_E in (2.1) is at most the right-hand side of (2.3). We give two proofs of this fact. The first one is a generalization of the proof of Theorem 1 in [9], whose by-product is needed in Section 4. Clearly, it is sufficient to prove an inequality of the type (2.1) for monic polynomials only. Thus, we assume that $p_k(z), 1 \le k \le m$, are all monic, so that $p(z)$ is monic too.

(i) For any $k = 1, \dots, m$, there exists $c_k \in \partial E$ such that

$$\|p_k\|_E = |p_k(c_k)|. \tag{5.22}$$

Applying Lemma 5.1 (or Lemma 2 of [9]) to the set $F = \{c_k\}_{k=1}^m$, we obtain that the function

$$u_m(z) := \max_{1\le k\le m} \log|z - c_k|, \qquad z \in \mathbb{C}, \tag{5.23}$$

is subharmonic in $\mathbb{C}$, and that

$$u_m(z) = \int \log|z-t|d\sigma_m(t), \qquad z \in \mathbb{C}, \tag{5.24}$$

where σ_m is a probability measure on $\mathbb{C}$. If Z_k is the set of zeros of $p_k(z)$ (counted according to multiplicities), $k = 1, \dots, m$, then

$$\begin{aligned} \sum_{k=1}^m \log\|p_k\|_E &= \sum_{k=1}^m \log|p_k(c_k)| = \sum_{k=1}^m \sum_{z\in Z_k} \log|c_k - z| \le \sum_{k=1}^m \sum_{z\in Z_k} u_m(z) \\ &= \sum_{z\in\bigcup_{k=1}^m Z_k} \int \log|z-t|d\sigma_m(t) = \int \log|p(t)|d\sigma_m(t). \end{aligned} \tag{5.25}$$

Using (5.16) and (5.17) of Lemma 5.2, we proceed further as follows:

$$\begin{aligned}
&\sum_{k=1}^{m} \log \|p_k\|_E \\
\le\ & \int \left(\log \|p\|_E + n \log \frac{1}{\operatorname{cap}(E)} + n \int \log |z-t| d\mu_E(z) \right) d\sigma_m(t) \\
=\ & \log \|p\|_E + n \log \frac{1}{\operatorname{cap}(E)} + n \int \int \log |z-t| d\mu_E(z) d\sigma_m(t) \\
=\ & \log \|p\|_E + n \log \frac{1}{\operatorname{cap}(E)} + n \int \int \log |z-t| d\sigma_m(t) d\mu_E(z) \\
=\ & \log \|p\|_E + n \log \frac{1}{\operatorname{cap}(E)} + n \int u_m(z) d\mu_E(z),
\end{aligned}$$

where we changed the order of integration by Fubini's theorem. It follows from the above estimate that

$$\prod_{k=1}^{m} \|p_k\|_E \le \left(\frac{\exp(\int u_m(z) d\mu_E(z))}{\operatorname{cap}(E)} \right)^n \|p\|_E. \tag{5.26}$$

Since $u_m(z) \le \log d_E(z)$ for any $z \in \mathbb{C}$, we immediately obtain that

$$M_E \le \frac{\exp(\int \log d_E(z) d\mu_E(z))}{\operatorname{cap}(E)}. \tag{5.27}$$

Note that we could proceed in a more direct way to prove (5.27), which is done below. However, the inequality (5.26) is needed for our analysis in Section 4.

(ii) Let $\{z_{k,n}\}_{k=1}^n$ be the zeros of $p(z)$ and let ν_n be the normalized zero counting measure for $p(z)$. Then, we use (5.1) with $F = E$, Fubini's theorem and Lemma 5.2 in the following estimate:

$$\begin{aligned}
&\frac{1}{n} \log \frac{\prod_{k=1}^m \|p_k\|_E}{\|p\|_E} \\
\le\ & \frac{1}{n} \log \frac{\prod_{k=1}^n \|z - z_{k,n}\|_E}{\|p\|_E} = \log \frac{1}{\|p\|_E^{1/n}} + \int \log d_E(z) d\nu_n(z) \\
=\ & \log \frac{1}{\|p\|_E^{1/n}} + \int \int \log |z-t| d\nu_n(z) d\sigma(t) = \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} d\sigma(t) \\
\le\ & \int g_\Omega(t, \infty) d\sigma(t) = \log \frac{1}{\operatorname{cap}(E)} + \int \int \log |z-t| d\sigma(t) d\mu_E(z) \\
=\ & \log \frac{1}{\operatorname{cap}(E)} + \int \log d_E(z) d\mu_E(z).
\end{aligned}$$

This completes the second proof of (5.27).

(iii) To show that equality holds in (5.27), we consider the n-th Fekete points $\{a_{k,n}\}_{k=1}^n, n \in \mathbb{N}$, for E, and define the Fekete polynomials as in (5.18):

$$F_n(z) = \prod_{k=1}^{n} (z - a_{k,n}), \quad n \in \mathbb{N}.$$

Observe that

$$\|z - a_{k,n}\|_E = d_E(a_{k,n}), \quad 1 \le k \le n, \quad n \in \mathbb{N}.$$

Since $\text{cap}(E) \neq 0$, the set E consists of more than one point and, therefore, $d_E(z)$ is a *strictly positive* continuous function in $\mathbb{C}$. Consequently, $\log d_E(z)$ is also continuous in $\mathbb{C}$, and we obtain by (5.21) of Lemma 5.3 that

$$\begin{aligned}(5.28)\qquad \lim_{n\to\infty}\left(\prod_{k=1}^n \|z-a_{k,n}\|_E\right)^{1/n} &= \lim_{n\to\infty}\exp\left(\frac{1}{n}\sum_{k=1}^n \log d_E(a_{k,n})\right)\\ = \exp\left(\lim_{n\to\infty}\int \log d_E(z)d\tau_n(z)\right) &= \exp\left(\int \log d_E(z)d\mu_E(z)\right).\end{aligned}$$

Finally, we have from the above and (5.20) that

$$M_E \geq \lim_{n\to\infty}\frac{\left(\prod_{k=1}^n \|z-a_{k,n}\|_E\right)^{1/n}}{\|F_n\|_E^{1/n}} = \frac{\exp(\int \log d_E(z)d\mu_E(z))}{\text{cap}(E)}.$$

□

Proof of Corollary 2.2. Since E is a bounded continuum, we obtain from Theorem 5.3.2(a) of [21, p. 138] that

$$\text{cap}(E) \geq \frac{\text{diam}(E)}{4}.$$

Thus, the corollary follows by combining this estimate with the obvious inequality

$$d_E(z) \leq \text{diam}(E), \quad z \in E,$$

and by using that $\mu_E(\mathbb{C}) = 1,\ \text{supp}\mu_E \subset E.$ □

Proof of Theorem 2.3. Let $D_R = \{z : |z| < R\}$ be a disk containing E and all zeros of the sequence $\{Q_n(z)\}_{n=1}^\infty$, where $R > 0$ is sufficiently large. Note that

$$\|Q_n\|_E^{1/n} \leq \left(\prod_{k=1}^n \|z-z_{k,n}\|_E\right)^{1/n} \leq 2R, \quad n \in \mathbb{N},$$

where $\{z_{k,n}\}_{k=1}^n$ are the zeros of $Q_n(z)$, as before. Consider a subsequence $N_1 \subset \mathbb{N}$ such that (see (2.7))

$$(5.29)\qquad \limsup_{n\to\infty}\|Q_n\|_E^{1/n} = \lim_{\substack{n\to\infty\\ n\in N_1}}\|Q_n\|_E^{1/n} =: C \geq \text{cap}(E).$$

Since the normalized counting measures ν_n for $Q_n(z)$, defined in (2.10), are supported on $\overline{D_R}$ for any $n \in \mathbb{N}$, we have by Helley's theorem (see [22]) that there exists a subsequence $N_2 \subset N_1$ satisfying

$$(5.30)\qquad \nu_n \xrightarrow{*} \nu, \quad \text{as } n\to\infty,\ n \in N_2 \subset N_1,$$

for a probability measure ν supported on $\overline{D_R}$. Using the continuity of $\log d_E(z)$ in $\mathbb{C}$, we obtain from (5.30) that

$$\begin{aligned}(5.31)\qquad \lim_{\substack{n\to\infty\\ n\in N_2}}\left(\prod_{k=1}^n \|z-z_{k,n}\|_E\right)^{1/n} &= \lim_{\substack{n\to\infty\\ n\in N_2}}\exp\left(\frac{1}{n}\sum_{k=1}^n \log d_E(z_{k,n})\right)\\ = \lim_{\substack{n\to\infty\\ n\in N_2}}\exp\left(\int \log d_E(z)d\nu_n(z)\right) &= \exp\left(\int \log d_E(z)d\nu(z)\right).\end{aligned}$$

Recall that $\{Q_n(z)\}_{n=1}^\infty$ is a sequence of extremal polynomials, so that

$$M_E = \frac{\exp(\int \log d_E(z) d\nu(z))}{C},$$

by (5.29) and (5.31). Hence we have by (2.3) that

$$\log \frac{1}{C} + \int \log d_E(z) d\nu(z) = \log \frac{1}{\operatorname{cap}(E)} + \int \log d_E(z) d\mu_E(z).$$

The integral representation (5.1) for $\log d_E(z)$ and Fubini's theorem give

$$\log \frac{1}{C} + \int \int \log |z-t| d\nu(z) d\sigma(t) = \log \frac{1}{\operatorname{cap}(E)} + \int \int \log |z-t| d\mu_E(z) d\sigma(t)$$

and

$$\int \left(\log \frac{1}{C} + \int \log |z-t| d\nu(z) \right) d\sigma(t) = \int \left(\log \frac{1}{\operatorname{cap}(E)} + \int \log |z-t| d\mu_E(z) \right) d\sigma(t). \tag{5.32}$$

On the other hand, we have by Lemma 5.2 that

$$\begin{aligned} \log \frac{1}{\|Q_n\|_E^{1/n}} + \int \log |z-t| d\nu_n(z) &\le g_\Omega(t,\infty) \\ &= \log \frac{1}{\operatorname{cap}(E)} + \int \log |z-t| d\mu_E(z), \quad t \in \mathbb{C}. \end{aligned}$$

Since $\operatorname{supp}\nu_n \subset \overline{D_R}$ for any $n \in \mathbb{N}$, we can pass to the limit in the above inequality, as $n \to \infty,\ n \in N_2$, and obtain by (5.29), (5.30) and the Lower Envelope Theorem (see Theorem I.6.9 in [22]) that

$$\log \frac{1}{C} + \int \log |z-t| d\nu(z) \le \log \frac{1}{\operatorname{cap}(E)} + \int \log |z-t| d\mu_E(z) \tag{5.33}$$

holds quasi everywhere (q.e.) in $\mathbb{C}$, i.e., with a possible exception of a set of zero capacity. Observe that $\operatorname{supp}\nu \subset \overline{D_R}$ and $\operatorname{supp}\mu_E \subset \overline{D_R}$, and that (5.33) holds μ_E-almost everywhere, because μ_E has finite energy. This implies that (5.33) holds for any $t \in \mathbb{C}$ by the Principle of Domination (see Theorem II.3.2 in [22]). Furthermore, the strict inequality is impossible in (5.33) for any $t \in (\mathbb{C} \setminus \overline{D_R}) \cap \operatorname{supp}\sigma$, as this would immediately violate (5.32), because both functions on the left and on the right of (5.33) are harmonic and continuous in $\mathbb{C} \setminus \overline{D_R}$. Thus, we have that

$$\log \frac{1}{C} + \int \log |z-t| d\nu(z) = \log \frac{1}{\operatorname{cap}(E)} + \int \log |z-t| d\mu_E(z), \tag{5.34}$$

for any $t \in (\mathbb{C} \setminus \overline{D_R}) \cap \operatorname{supp}\sigma$. It follows from (5.34) and Theorem 3.1.2 of [21, p. 53] that

$$\log \frac{\operatorname{cap}(E)}{C} = \lim_{\substack{t\to\infty \\ t \in \operatorname{supp}\sigma}} \left(\int \log |z-t| d\mu_E(z) - \int \log |z-t| d\nu(z) \right) = 0,$$

where we can pass to the above limit because $\operatorname{supp}\sigma$ is unbounded by Lemma 5.1. Since (5.29) holds with $C = \operatorname{cap}(E)$, by the above proof, then (2.9) now follows from (2.7). ∎

Proof of Theorem 2.4. Note that any of the assumptions (i) or (ii) implies that $\mathrm{cap}(E) > 0$. Assume that $\{Q_n(z)\}_{n=1}^{\infty}$ is a sequence of monic asymptotically extremal polynomials. On taking log of (2.5) and using (2.3), we have

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \log \|z - z_{k,n}\|_E + \log \frac{1}{\|Q_n\|_E}\right) = \int \log d_E(z) d\mu_E(z) + \log \frac{1}{\mathrm{cap}(E)}$$

or

$$\lim_{n\to\infty} \left(\int \log d_E(z) d\nu_n(z) + \frac{1}{n} \log \frac{1}{\|Q_n\|_E}\right) = \int \log d_E(z) d\mu_E(z) + \log \frac{1}{\mathrm{cap}(E).}$$

It follows from (5.1)-(5.2) and Fubini's theorem that

$$\lim_{n\to\infty} \int \left(\int \log |z - t| d\nu_n(z) + \frac{1}{n} \log \frac{1}{\|Q_n\|_E}\right) d\sigma(t)$$
$$= \int \left(\int \log |z - t| d\mu_E(z) + \log \frac{1}{\mathrm{cap}(E)}\right) d\sigma(t) = \int g_\Omega(t, \infty) d\sigma(t),$$

where we used (5.17) on the last step. The above equation can be also written as

$$\lim_{n\to\infty} \int \left(\frac{1}{n} \log \frac{|Q_n(t)|}{\|Q_n\|_E} - g_\Omega(t, \infty)\right) d\sigma(t) = 0. \tag{5.35}$$

Recall that, for any $n \in \mathbb{N}$,

$$\frac{1}{n} \log \frac{|Q_n(t)|}{\|Q_n\|_E} - g_\Omega(t, \infty) \le 0, \quad t \in \mathbb{C}, \tag{5.36}$$

by (5.16). Hence, for any disk $D_r(z) := \{t : |z - t| < r\}$, $z \in \mathrm{supp}\sigma$, we have

$$\liminf_{n\to\infty} \int_{D_r(z)} \left(\frac{1}{n} \log \frac{|Q_n(t)|}{\|Q_n\|_E} - g_\Omega(t, \infty)\right) d\sigma(t) \ge 0,$$

as the opposite inequality would violate (5.35), because of (5.36). It follows that

$$\liminf_{n\to\infty} \sup_{D_r(z)\cap \mathrm{supp}\sigma} \left(\frac{1}{n} \log \frac{|Q_n(t)|}{\|Q_n\|_E} - g_\Omega(t, \infty)\right) \ge 0, \quad z \in \mathrm{supp}\sigma. \tag{5.37}$$

Note that for the polynomials $P_n(z) := Q_n(z)/\|Q_n\|_E$, $n \in \mathbb{N}$, we have

$$\|P_n\|_E = 1, \quad n \in \mathbb{N}. \tag{5.38}$$

If E has empty interior and connected and regular complement, then (2.12) now follows from Theorem 1 of [14], (5.37) and (5.38). This proves Theorem 2.4 in the case (i).

However, the case (ii) requires an additional argument to show that

$$\lim_{n\to\infty} \nu_n(B) = 0, \tag{5.39}$$

for any compact set $B \subset \mathbb{C} \setminus \overline{\Omega}$. Indeed, if we apply Theorem 1 of [14] to the sequence $\{P_n(z)\}_{n=1}^{\infty}$ on $\tilde{E} := \mathbb{C} \setminus \Omega$ in this case, then (2.12) is implied by (5.37), (5.38) and (5.39). Thus, our current goal is to prove (5.39). Observe that the open set $\tilde{G} := \mathbb{C} \setminus \overline{\Omega}$ is a simply connected domain, which contains the original domain G. Clearly, $\tilde{G}$ is the interior of $\tilde{E}$. Note that

$$g_\Omega(t,\infty) = 0, \quad t \in \tilde{G},$$

by (5.17) and Theorem III.14 of [24, p. 61]. Lemma 5.1 gives by (5.3) that there exists a disk $D_r(\zeta) \subset G \subset \tilde{G}$ such that $S := \overline{D_r(\zeta)} \cap \operatorname{supp}\sigma \neq \emptyset$, $\zeta \in \operatorname{supp}\sigma$, and that

$$\liminf_{n\to\infty} \sup_{t\in S} \left(\frac{1}{n} \log \frac{|Q_n(t)|}{\|Q_n\|_E}\right) \geq 0, \tag{5.40}$$

by (5.37). One can see that (5.40) holds with S replaced by any closed disk $K \subset \tilde{G}$ such that $K \cap S = \emptyset$. Indeed, if this is not true, then we define a harmonic function $u(z)$ in $\tilde{G} \setminus K$, with boundary values

$$u(t) = \begin{cases} 0, \quad t \in \partial\tilde{G}, \\ \liminf\limits_{n\to\infty} \sup\limits_{t\in K} \left(\dfrac{1}{n} \log \dfrac{|Q_n(t)|}{\|Q_n\|_E}\right) < 0, \quad t \in \partial K. \end{cases}$$

Using the maximum principle for subharmonic functions, we obtain

$$\liminf_{n\to\infty} \sup_{z\in S} \left(\frac{1}{n} \log \frac{|Q_n(z)|}{\|Q_n\|_E}\right) \leq \liminf_{n\to\infty} \sup_{z\in S} u(z) < 0,$$

which contradicts (5.40). Thus, we can consider any compact set $B \subset \tilde{G}$, assuming for the proof of (5.39) that $B \cap S = \emptyset$. Define the subharmonic in $\tilde{G}$ function

$$h_n(z) := \frac{1}{n} \log \frac{|Q_n(z)|}{\|Q_n\|_E} + \frac{1}{n} \sum_{z_{k,n}\in B} g_{\tilde{G}}(z, z_{k,n}), \quad z \in \tilde{G},$$

where the sum is over all zeros $z_{k,n}$ of $Q_n(z)$ in B, and where $g_{\tilde{G}}(z, z_{k,n})$ is the Green function of $\tilde{G}$ with pole at $z_{k,n}$. It follows by the maximum principle that

$$\limsup_{n\to\infty} \sup_{z\in S} h_n(z) \leq \limsup_{n\to\infty} \sup_{z\in\partial\tilde{G}} h_n(z) = 0, \tag{5.41}$$

because $g_{\tilde{G}}(z, z_{k,n}) = 0$ for $z \in \partial\tilde{G}$, as $\tilde{G}$ is simply connected and regular. We continue by estimating

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \inf_{z\in S} \left(\frac{1}{n} \sum_{z_{k,n}\in B} g_{\tilde{G}}(z, z_{k,n})\right) \\ &= \limsup_{n\to\infty} \inf_{z\in S} \left(h_n(z) - \frac{1}{n} \log \frac{|Q_n(z)|}{\|Q_n\|_E}\right) \\ &\leq \limsup_{n\to\infty} \left(\sup_{z\in S} h_n(z) - \sup_{z\in S} \left(\frac{1}{n} \log \frac{|Q_n(z)|}{\|Q_n\|_E}\right)\right) \\ &\leq \limsup_{n\to\infty} \sup_{z\in S} h_n(z) - \liminf_{n\to\infty} \sup_{z\in S} \left(\frac{1}{n} \log \frac{|Q_n(z)|}{\|Q_n\|_E}\right) \leq 0, \end{aligned}$$

where the last inequality follows from (5.40) and (5.41). Hence,

$$\lim_{n\to\infty} \inf_{z\in S} \left(\frac{1}{n} \sum_{z_{k,n}\in B} g_{\tilde{G}}(z, z_{k,n})\right) = 0. \tag{5.42}$$

Since $B \cap S = \emptyset$, we have

$$a := \inf_{z \in S} \inf_{t \in B} g_{\tilde{G}}(z,t) > 0.$$

This inequality and (5.42) yield

$$0 = \lim_{n\to\infty} \inf_{z\in S} \left(\frac{1}{n} \sum_{z_{k,n} \in B} g_{\tilde{G}}(z, z_{k,n}) \right) \geq a \limsup_{n\to\infty} \nu_n(B),$$

so that (5.39) is proved. Thus, the proof of the case (ii) is also completed. □

Proof of Theorem 2.5. Let $D_R := \{z : |z| < R\}$ be large enough, so that $E \subset D_R$. Since (2.12) is valid, we have that there are only $o(n)$ zeros of $Q_n(z)$ outside of D_R, as $n \to \infty$. Assume that $Q_n(z)$ is monic for all $n \in \mathbb{N}$ and define a monic polynomial $\tilde{Q}_n(z)$, whose zeros are the zeros of $Q_n(z)$ contained in D_R. We denote the set of zeros of $\tilde{Q}_n(z)$ by $\tilde{Z}_n$ and consider the zero counting measures for $\tilde{Q}_n(z)$:

$$\lambda_n := \frac{1}{n} \sum_{z_{k,n} \in \tilde{Z}_n} \delta_{z_{k,n}}, \quad n \in \mathbb{N}.$$

It is clear from (2.12) that

$$\lambda_n \xrightarrow{*} \mu_E, \quad \text{as } n \to \infty. \tag{5.43}$$

Observe that $\text{supp}\lambda_n \subset \overline{D_R},\ n \in \mathbb{N}$, so that we obtain

$$\begin{aligned} \lim_{n\to\infty} \left(\prod_{z_{k,n}\in\tilde{Z}} \|z - z_{k,n}\|_E \right)^{1/n} &= \lim_{n\to\infty} \exp\left(\frac{1}{n} \sum_{z_{k,n}\in\tilde{Z}} \log d_E(z_{k,n}) \right) \\ = \lim_{n\to\infty} \exp\left(\int \log d_E(z) d\lambda_n(z) \right) &= \exp\left(\int \log d_E(z) d\mu_E(z) \right), \end{aligned} \tag{5.44}$$

by the weak* convergence in (5.43). Moreover, Theorem 2.1 of [6] and (5.43) imply that

$$\lim_{n\to\infty} \|\tilde{Q}_n\|_E^{1/n} = \text{cap}(E), \tag{5.45}$$

because Ω is regular by our assumptions and all zeros of $\tilde{Q}_n(z)$ are uniformly bounded. Since the norm of product is at most the product of norms, we have

$$\begin{aligned} \liminf_{n\to\infty} \left(\frac{\prod_{k=1}^n \|z - z_{k,n}\|_E}{\|Q_n\|_E} \right)^{1/n} &\geq \liminf_{n\to\infty} \left(\frac{\prod_{z_{k,n}\in\tilde{Z}_n} \|z - z_{k,n}\|_E}{\|\tilde{Q}_n\|_E} \right)^{1/n} \\ &= \frac{\exp(\int \log d_E(z) d\mu_E(z))}{\text{cap}(E)}, \end{aligned}$$

by (5.44) and (5.45). The opposite inequality follows from Theorem 2.1:

$$\limsup_{n\to\infty} \left(\frac{\prod_{k=1}^n \|z - z_{k,n}\|_E}{\|Q_n\|_E} \right)^{1/n} \leq \frac{\exp(\int \log d_E(z) d\mu_E(z))}{\text{cap}(E)}.$$

Hence, (2.5) holds true for the sequence $\{Q_n(z)\}_{n=1}^\infty$ and the proof is finished. □

Proof of Corollary 3.3. Observe that any Borel set $B \subset \gamma$ has two images $\Phi_+(B)$ and $\Phi_-(B)$ on the unit circle. Thus, we have for the equilibrium measure μ_γ that

$$\mu_\gamma(B) = m(\Phi_+(B)) + m(\Phi_-(B)),$$

where $dm = |dw|/(2\pi)$ is the normalized arclength on $\{|w| = 1\}$, by (3.14). It follows that

$$\mu_\gamma = \frac{1}{2\pi}\left(|\Phi_+'(e^{i\theta})| + |\Phi_-'(e^{i\theta})|\right) d\theta, \quad e^{i\theta} \in \gamma,$$

where one finds by a straightforward calculation that

$$|\Phi_\pm'(e^{i\theta})| = \frac{|\Phi_\pm(e^{i\theta}) + \sin(\alpha/4)|}{\sqrt{2(\cos\theta - \cos(\alpha/2))}}, \quad e^{i\theta} \in \gamma.$$

Hence, (3.11) is proved. If $\alpha \le \pi$ then

$$d_\gamma(e^{i\theta}) = |e^{i\theta} - e^{-i\alpha/2}| = 2\sin(\theta/2 + \alpha/4), \quad 0 \le \theta \le \alpha/2.$$

However, if $\pi \le \alpha \le 2\pi$ then

$$d_\gamma(e^{i\theta}) = \begin{cases} 2\sin(\theta/2 + \alpha/4), & 0 \le \theta \le \pi - \alpha/2, \\ 2, & \pi - \alpha/2 \le \theta \le \alpha/2. \end{cases}$$

Using the symmetry of γ, we obtain (3.12) from (2.3), (3.7), (3.11) and the above formulae for $d_\gamma(e^{i\theta})$. The second part of Corollary 3.3 follows from Theorems 2.4 and 2.5. □

Proof of Theorem 4.1. We shall adjust part (iii) of the proof of Theorem 2.1, to show that (4.5)-(4.6) are true. For the n-th Fekete points $\{a_{k,n}\}_{k=1}^n \subset \partial E,\ n \in \mathbb{N}$, consider the Fekete polynomials (see (5.18))

$$F_n(z) = \prod_{k=1}^n (z - a_{k,n}), \quad n \in \mathbb{N}.$$

A proper factoring in (4.5) can be achieved by grouping Fekete points as follows. We define a subset $\mathcal{F}_{l,n} \subset \{a_{k,n}\}_{k=1}^n$, associated with each point $\zeta_l,\ l = 1, \ldots, s$, so that $a_{k,n} \in \mathcal{F}_{l,n}$ if

$$d_E(a_{k,n}) = |a_{k,n} - \zeta_l|, \quad 1 \le k \le n. \tag{5.46}$$

In the case that (5.46) holds for more than one ζ_l, we refer $a_{k,n}$ to only one set $\mathcal{F}_{l,n}$, to avoid an overlap of these sets. It is clear from (4.4) that, for any $n \in \mathbb{N}$,

$$\bigcup_{l=1}^s \mathcal{F}_{l,n} = \{a_{k,n}\}_{k=1}^n \quad \text{and} \quad \mathcal{F}_{l_1,n} \cap \mathcal{F}_{l_2,n} = \emptyset, \qquad l_1 \ne l_2.$$

The desired factors of $F_n(z)$ in (4.5) are defined as follows:

$$F_{l,n}(z) := \prod_{a_{k,n} \in \mathcal{F}_{l,n}} (z - a_{k,n}), \quad l = 1, \ldots, s. \tag{5.47}$$

If $m > s$ then we let $F_{l,n}(z) \equiv 1$ for $l = s+1, \ldots, m$. Using (5.46), we obtain that

$$\|F_{l,n}\|_E = \prod_{a_{k,n} \in \mathcal{F}_{l,n}} |\zeta_l - a_{k,n}| = \prod_{a_{k,n} \in \mathcal{F}_{l,n}} d_E(a_{k,n}), \quad l = 1, \ldots, s,$$

which gives by (5.28) that

$$\lim_{n\to\infty}\left(\prod_{l=1}^{m}\|F_{l,n}\|_E\right)^{1/n} = \lim_{n\to\infty}\left(\prod_{k=1}^{n} d_E(a_{k,n})\right)^{1/n} = \exp\left(\int \log d_E(z)d\mu_E(z)\right).$$

Hence, (4.6) follows by combining (5.20) with the above equation. □

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